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Phase-function method for Coulomb-distorted nuclear scattering†

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Abstract. The phase-function method is very effective in treating quantum mechanical scattering problems for short-range local potentials. We adapt the phase method to deal with Coluomb plus Graz non-local separable potentials and derive a closed-form expression for the scattering phase shift. Our approach to the problem circumvents in a rather natural way the typical difficulties of incorporating the Coulomb interaction in a nuclear phase-shift calculation. We demonstrate the usefulness of our constructed expression by means of a model calculation.

1. Introduction

The phase-function method represents an efficient approach [1] to evaluating the scattering phase shifts for quantum mechanical problems and is based on the separation of the radial wavefunction of the Schrödinger equation into an amplitude part $\alpha_l(k, r)$ and an oscillating part with a variable phase $\delta_l(k, r)$. Physically, this amounts to factorising out the two effects of the potential which manifest themselves in deforming the wavefunction and in producing the scattering phases [2]. The function $\delta_l(k, r)$, called the phase function, has at each point the meaning of the phase shift of the wavefunction for scattering by the potential truncated at a distance r. A completely amputated potential will not produce any phase shift. Thus $\delta_l(k, 0) = 0$. For a local potential $\delta_l(k, r)$ satisfies a first-order non-linear differential equation. The scattering phase shift $\delta_l(k)$ is obtained by solving this equation from the origin to the asymptotic region with the initial condition $\delta_l(k, 0) = 0$. During the solution of the phase equation, $\delta_l(k, r)$ is built up by the potential as one moves away from the origin and it reaches its asymptotic value as soon as one gets out of the range of the potential. Obviously, $\delta_l(k) = \lim_{r\to\infty} \delta_l(k, r)$. Once the phase function is known determination of the amplitude function reduces to a trivial problem.

In contrast to the local case, the phase function $\delta_l(k, r)$ for a non-local potential does not have a simple physical meaning with regard to the accumulation of phases. A non-local potential couples the wavefunction at one point with its values at all other points. This implies that accumulation of phase depends on the wavefunction for all values of r. Therefore, it is of considerable interest to explore the possibility of extending the phase approach to the case of scattering on non-local potentials. For general non-local potentials the phase equation has a very complicated mathematical structure

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[3]. In the special case of a separable potential, however, the phase function can be written in closed form without solving the phase equation. This can be seen as follows.

The radial Schrödinger equation for a rank-N separable potential is given by

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-l(l+1)r^{-2}\right)U_l(k,r)=\sum_{i=1}^{N}\lambda_l^{(i)}f_l^{(i)}(r)\int_0^\infty \mathrm{d}s\, g_l^{(i)}(s)U_l(k,s) \tag{1}$$

where $f_l^{(i)}(r)$ and $g_l^{(i)}(r)$ are the form factors of the potential and $\lambda_l^{(i)}$ is the strength parameter. Here the two-body centre of mass energy $E=k^2>0$. We work in units in which $\hbar^2/2m$ is unity. We now convert (1) to an integral equation with the help of the Green function

$$G_l^{(R)}(r,r')=\begin{cases}-k^{-1}(\hat{j}_l(kr)\hat{\eta}_l(kr')-\hat{\eta}_l(kr)\hat{j}_l(kr')) & r'<r\\ 0 & r'>r\end{cases} \tag{2}$$

satisfying a regular boundary condition. This gives

$$U_l(k,r)=\hat{j}_l(kr)-k^{-1}\sum_{i=1}^{N}\lambda_l^{(i)}\,d_l^{(i)}(k)\int_0^r \mathrm{d}r'\,(\hat{j}_l(kr)\hat{\eta}_l(kr')-\hat{\eta}_l(kr)\hat{j}_l(kr'))f_l^{(i)}(r') \tag{3}$$

with

$$d_l^{(i)}(k)=\int_0^\infty \mathrm{d}r\, g_l^{(i)}(r)U_l(k,r). \tag{4}$$

For the Riccati–Bessel functions $\hat{j}_l(kr)$ and $\hat{\eta}_l(kr)$ we shall follow the phase convention of Calogero [1] with the Hankel function of the first kind written as $\hat{h}_l^{(1)}(x)=-\hat{\eta}_l(x)+\mathrm{i}\hat{j}_l(x)$. Introducing the phase and amplitude functions by

$$\alpha_l(k,r)\cos\delta_l(k,r)=1-k^{-1}\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)}(k)\int_0^r \mathrm{d}r'\,\hat{\eta}_l(kr')f_l^{(i)}(r') \tag{4a}$$

and

$$\alpha_l(k,r)\sin\delta_l(k,r)=-k^{-1}\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)}(k)\int_0^r \mathrm{d}r'\,\hat{j}_l(kr')f_l^{(i)}(r') \tag{4b}$$

then (3) can be expressed as

$$U_l(k,r)=\alpha_l(k,r)(\cos\delta_l(k,r)\hat{j}_l(kr)-\sin\delta_l(k,r)\hat{\eta}_l(kr)). \tag{5}$$

From (4a) and (4b) we have

$$\tan\delta_l(k,r)=-\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)}(k)\int_0^r \mathrm{d}r'\hat{j}_l(kr')f_l^{(i)}(r')$$
$$\times\left(k-\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)}(k)\int_0^r \mathrm{d}r'\hat{\eta}_l(kr')f_l^{(i)}(r')\right)^{-1}. \tag{6}$$

Expression (6) for the phase function involves the unknown quantity $d_l^{(i)}(k)$. Since (3) represents an inhomogenous integral equation with a degenerate kernel, it can be solved to write

$$d_l^{(i)}(k)=\frac{1}{\det_N A_l(k)}\sum_{j=1}^{N}a_l^{(i,j)}(k)\,Y_l^{(j)}(k). \tag{7}$$

The elements of the Fredholm determinant $\det_N A_l(k)$ are

$$A_l^{(i,j)}(k)=\delta_{ij}+k^{-1}\lambda_l^{(j)}\int_0^\infty\int_0^r \mathrm{d}r'\,\mathrm{d}r\,g_l^{(j)}(r)\,\{\hat{j}_l(kr)\hat{\eta}_l(kr')-\hat{\eta}_l(kr)\hat{j}_l(kr')\}f_l^{(i)}(r'). \tag{8}$$

The quantities $a_l^{(i,j)}(k)$ stand for the cofactors of the $A_l^{(i,j)}(k)$. We also define

$$Y_l^{(j)}(k)=\int_0^\infty \mathrm{d}r\,\hat{j}_l(kr)g_l^{(j)}(r). \tag{9}$$

Using (7) in (6) we obtain

$$\tan\delta_l(k,r)=-\sum_{i,j=1}^{N}\lambda_l^{(i)}a_l^{(i,j)}(k)\,Y_l^{(j)}(k)\int_0^r \mathrm{d}r'\,\hat{j}_l(kr')f_l^{(i)}(r')$$
$$\times\left(k\,\det_N A_l(k)-\sum_{i,j=1}^{N}\lambda_l^{(i)}a_l^{(i,j)}(k)\,Y_l^{(j)}(k)\int_0^r \mathrm{d}r'\,\hat{\eta}_l(kr')f_l^{(i)}(r')\right)^{-1}. \tag{10}$$

In the limit $r\to\infty$, (10) gives the scattering phase shift in terms of the transforms of the regular and irregular solutions of the free-particle Schrödinger equation by the form factors of the potential and one does not require solving the phase equation to get the phase shift. In general, the forms of $f_l^{(i)}(r)$ and $g_l^{(i)}(r)$ are quite simple for most of the separable representations of the nucleon–nucleon interaction so that integrals like $\int_0^\infty \mathrm{d}r\,\hat{j}_l(kr)g_l^{(i)}(r)$, $\int_0^\infty \mathrm{d}r\,\hat{\eta}_l(kr)f_l^{(i)}(r)$, etc, can be expressed in closed form.

In the non-relativistic model for proton–proton scattering one would add a repulsive Coulomb interaction to the separable potential considered above. The object of the present paper is to treat the Coulomb-distorted nuclear scattering within the framework of the phase method. The calculation of the phase function or the phase shift for such a problem need not start with the kinetic energy as a zero-order Hamiltonian. Instead, one should begin by calculating the states, Green function, etc, for a model Hamiltonian that involves the Coulomb interaction. In § 2 we deal with the Coulomb plus rank-N separable potential. We present a case study in § 3 and derive a closed form expression for the $^1S_0$ proton–proton scattering phase shift. Our derivation based on the phase method is quite straightforward and does not rely on the expression for the off-shell t matrix and its subsequent adaptation on the energy shell [4]. The final result is given in terms of Gaussian hypergeometric functions of complex arguments. We demonstrate the usefulness of our expression in § 4 by calculating numbers for p–p scattering phase shifts.

2. Phase method for Coulomb plus separable potentials

For the Coulomb-distorted rank-N separable potential the Schrödinger equation in (1) is modified as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-l(l+1)r^{-2}-2k\eta r^{-1}\right)U_l^{\mathrm{cs}}(k,r)=\sum_{i=1}^{N}\lambda_l^{(i)}f_l^{(i)}(r)\int_0^\infty \mathrm{d}t\,g_l^{(i)}(t)\,U_l^{\mathrm{cs}}(k,t). \tag{11}$$

Here η is the so-called Sommerfeld parameter. Henceforth, we shall use superscripts C and CS for quantities related to pure Coulomb and Coulomb plus separable potentials respectively. The integral form of (11) is

$$U_l^{\mathrm{CS}}(k,r)=k\phi_l^{\mathrm{C}}(k,r)+\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)\mathrm{CS}}(k)\int_0^r \mathrm{d}r'\,f_l^{(i)}(r')G_l^{\mathrm{C(R)}}(r,r') \tag{12}$$

with

$$d_l^{(i)\mathrm{CS}}(k)=\int_0^\infty \mathrm{d}t\, g_l^{(i)}(t)\, U_l^{\mathrm{CS}}(k,t) \tag{13}$$

and $\phi_l^{\mathrm{C}}(k,r)$, the Coulomb regular solution, written as

$$\phi_l^{\mathrm{C}}(k,r)=r^{l+1}\,\mathrm{e}^{\mathrm{i}kr}\Phi(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr). \tag{14}$$

For usage in the phase method we write the regular Coulomb Green function $G_l^{\mathrm{C(R)}}(r,r')$ in the form

$$G_l^{\mathrm{C(R)}}(r,r')=|\mathscr{F}_l^{\mathrm{C}}(k)|^{-2}[\phi_l^{\mathrm{C}}(k,r)\tfrac{1}{2}(\mathscr{F}_l^{\mathrm{C}}(-k)f_l^{\mathrm{C}}(k,r')+\mathscr{F}_l^{\mathrm{C}}(k)f_l^{\mathrm{C}}(-k,r'))$$
$$-\phi_l^{\mathrm{C}}(k,r')\tfrac{1}{2}(\mathscr{F}_l^{\mathrm{C}}(-k)f_l^{\mathrm{C}}(k,r)+\mathscr{F}_l^{\mathrm{C}}(k)f_l^{\mathrm{C}}(-k,r'))] \tag{15}$$

for $r'<r$ and zero elsewhere.

The irregular Coulomb function is

$$f_l^{\mathrm{C}}(k,r)=-(2\mathrm{i}kr)^{l+1}\,\mathrm{e}^{\pi\eta/2}\,\mathrm{e}^{\mathrm{i}kr}\Psi(l+1+\mathrm{i}\eta,2l+2;-2\mathrm{i}kr) \tag{16}$$

with $f_l^{\mathrm{C}}(-k,r)=f_l^{\mathrm{C}*}(k,r)$. As with the irregular solution, the Coulomb Jost function

$$\mathscr{F}_l^{\mathrm{C}}(k)=\frac{(2k)^{-l}\,\mathrm{e}^{\pi\eta/2+\mathrm{i}l\pi/2}\Gamma(2l+2)}{\Gamma(l+1+\mathrm{i}\eta)} \tag{17}$$

also satisfies $\mathscr{F}_l^{\mathrm{C}}(-k)=\mathscr{F}_l^{\mathrm{C}*}(k)$. In (14) and (16) $\Phi(\ )$ and $\Psi(\ )$ stand for the regular and irregular confluent hypergeometric functions. From (12) and (15) we write

$$U_l^{\mathrm{CS}}(k,r)=\alpha_l^{\mathrm{CS}}(k,r)[k\phi_l^{\mathrm{C}}(k,r)\cos\delta_l^{\mathrm{CS}}(k,r)$$
$$+\tfrac{1}{2}(\mathscr{F}_l^{\mathrm{C}}(-k)f_l^{\mathrm{C}}(k,r)+\mathscr{F}_l^{\mathrm{C}}(k)f_l^{\mathrm{C}}(-k,r))\sin\delta_l^{\mathrm{CS}}(k,r)]. \tag{18}$$

In deriving (18) we have used

$$\alpha_l^{\mathrm{CS}}(k,r)\cos\delta_l^{\mathrm{CS}}(k,r)$$
$$=1+k^{-1}|\mathscr{F}_l^{\mathrm{C}}(k)|^{-2}\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)\mathrm{CS}}(k)\int_0^r \mathrm{d}r'\,f_l^{(i)}(r')$$
$$\times\tfrac{1}{2}(\mathscr{F}_l^{\mathrm{C}}(-k)f_l^{\mathrm{C}}(k,r')+\mathscr{F}_l^{\mathrm{C}}(k)f_l^{\mathrm{C}}(-k,r')) \tag{19a}$$

and

$$\alpha_l^{\mathrm{CS}}(k,r)\sin\delta_l^{\mathrm{CS}}(k,r)=-|\mathscr{F}_l^{\mathrm{C}}(k)|^{-2}\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)\mathrm{CS}}(k)\int_0^r f_l^{(i)}(r')\phi_l^{\mathrm{C}}(k,r')\,\mathrm{d}r'. \tag{19b}$$

Equations (19a) and (19b) can be combined to obtain

$$\tan\delta_l^{\mathrm{CS}}(k,r)=-\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)\mathrm{CS}}(k)\int_0^r f_l^{(i)}(r')\phi_l^{\mathrm{C}}(k,r')\,\mathrm{d}r'$$
$$\times\left(|\mathscr{F}_l^{\mathrm{C}}(k)|^2+k^{-1}\sum_{i=1}^{N}\lambda_l^{(i)}d_l^{(i)\mathrm{CS}}(k)\int_0^r f_l^{(i)}(r')\tfrac{1}{2}(\mathscr{F}_l^{\mathrm{C}}(-k)f_l^{\mathrm{C}}(k,r')\right.$$
$$\left.+\mathscr{F}_l^{\mathrm{C}}(k)f_l^{\mathrm{C}}(-k,r'))\,\mathrm{d}r'\right)^{-1}. \tag{20}$$

From the integral equation defined by (12), (13) and (15) we have obtained $d_l^{(i)\text{CS}}(k)$ in the form of (7). More specifically, we now have

$$d_l^{(i)\text{CS}}(k)=\frac{1}{\det_N A_l^{\text{CS}}(k)}\sum_{j=1}^{N} a_l^{(i,j)\text{CS}}(k)\,Y_l^{(j)\text{CS}}(k). \tag{21}$$

In analogy with (8) and (9)

$$A_l^{(i,j)\text{CS}}(k)=\delta_{ij}-\lambda_l^{(j)}\int_0^\infty\int_0^r \mathrm{d}r\,\mathrm{d}r'\,g_l^{(j)}(r)G_l^{\text{C(R)}}(r,r')f_l^{(i)}(r') \tag{22}$$

and

$$Y_l^{(j)\text{CS}}(k)=k\int_0^\infty \mathrm{d}r\,\phi_l^{\text{C}}(k,r)g_l^{(j)}(r). \tag{23}$$

Understandably, the $a_l^{(i,j)\text{CS}}(k)$ will now stand for the cofactors of the $A_l^{(i,j)\text{CS}}(k)$. From (20)-(23) we get

$$\begin{aligned}\tan\delta_l^{\text{CS}}(k,r)=&-\sum_{i,j=1}^{N}\lambda_l^{(i)}a_l^{(i,j)\text{CS}}(k)\,Y_l^{(j)\text{CS}}(k)\int_0^r \mathrm{d}r'\,\phi_l^{\text{C}}(k,r')f_l^{(i)}(r')\\ &\times\bigg(|\mathscr{F}_l^{\text{C}}(k)|^2\det_N A_l^{\text{CS}}(k)+(2k)^{-1}\sum_{i,j=1}^{N}\lambda_l^{(i)}a_l^{(i,j)\text{CS}}(k)\,Y_l^{(j)\text{CS}}(k)\\ &\times\int_0^r \mathrm{d}r'\,f_l^{(i)}(r')(\mathscr{F}_l^{\text{C}}(-k)f_l^{\text{C}}(k,r')+\mathscr{F}_l^{\text{C}}(k)f_l^{\text{C}}(-k,r'))\bigg)^{-1}.\end{aligned} \tag{24}$$

We have verified that when the Coulomb potential is turned off by putting $\eta=0$ the result in (24) goes over to that given in (10) for the phase function induced by the separable potential alone. Thus, in the limit $r\to\infty$, (24) will give the scattering phase shift for the Coulomb plus rank-N separable potentials in terms of the transforms of the regular and irregular Coulomb solutions by the form factors of the potential. In the next section we specialise to the s-wave case, omit the subscript $l=0$, and obtain $\tan\delta^{\text{CS}}(k)$ for the Coulumb plus Graz-I potential [5] in closed analytic form.

3. A case study

In the recent past the Graz group obtained a realistic fit to the N-N interaction in terms of a separable potential for which $f_l^{(i)}(r)=g_l^{(i)}(r)$ $(=v_l^{(i)}(r)$ say). For p-p scattering in the $^1\text{S}_0$ channel the Graz-I potential [5] is given by

$$V(r,r')=\lambda^{(1)}v^{(1)}(r)v^{(1)}(r')+\lambda^{(2)}v^{(2)}(r)v^{(2)}(r') \tag{25}$$

with

$$v^{(1)}(r)=\mathrm{e}^{-\alpha r} \tag{26a}$$

and

$$v^{(2)}(r)=(1-\tfrac{1}{2}\beta r)\mathrm{e}^{-\beta r}. \tag{26b}$$

From (24) $\tan \delta^{\mathrm{CS}}(k)$ for the potential in (25) is obtained in the form

$$\tan \delta^{\mathrm{CS}}(k) = -\sum_{i,j=1}^{2} \lambda^{(i)} a^{(i,j)\mathrm{CS}}(k)\, Y^{(i)\mathrm{CS}}(k)\, Y^{(j)\mathrm{CS}}(k) \times\left(k|\mathscr{F}^{\mathrm{C}}(k)|^2 \det{}_2 A^{\mathrm{CS}}(k) + \sum_{i,j=1}^{2} \lambda^{(i)} a^{(i,j)\mathrm{CS}}(k)\, Y^{(j)\mathrm{CS}}(k) X^{(i)\mathrm{CS}}(k)\right)^{-1} \tag{27}$$

where

$$X^{(i)\mathrm{CS}}(k) = \tfrac{1}{2}\int_0^\infty \mathrm{d}r'\, v^{(i)}(r')(\mathscr{F}^{\mathrm{C}}(-k) f^{\mathrm{C}}(k, r') + \mathscr{F}^{\mathrm{C}}(k) f^{\mathrm{C}}(-k, r')). \tag{28}$$

Using the form factors in (26a) and (26b) we have found the following results:

$$X^{(1)\mathrm{CS}}(k) = \mathrm{e}^{\pi\eta/2}\,\mathrm{Re}\left[\frac{\mathscr{F}^{\mathrm{C}}(-k)}{(\alpha - \mathrm{i}k)\Gamma(2+\mathrm{i}\eta)}\, {}_2F_1\left(1, \mathrm{i}\eta; 2+\mathrm{i}\eta; \frac{\alpha+\mathrm{i}k}{\alpha-\mathrm{i}k}\right)\right] \tag{29a}$$

$$X^{(2)\mathrm{CS}}(k) = \mathrm{e}^{\pi\eta/2}\,\mathrm{Re}\left\{\frac{\mathscr{F}^{\mathrm{C}}(-k)}{\Gamma(2+\mathrm{i}\eta)}\left[\frac{\beta(1+\mathrm{i}\eta)}{2(\beta^2+k^2)} + \frac{1}{(\beta-\mathrm{i}k)}\left(1 - \frac{\beta}{2(\beta-\mathrm{i}k)} + \frac{\beta(\mathrm{i}k - 2k\eta - \beta)}{2(\beta^2+k^2)}\right) {}_2F_1\left(1, \mathrm{i}\eta; 2+\mathrm{i}\eta; \frac{\beta+\mathrm{i}k}{\beta-\mathrm{i}k}\right)\right]\right\} \tag{29b}$$

$$Y^{(1)\mathrm{CS}}(k) = \frac{k}{(\alpha^2+k^2)}\left(\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)^{\mathrm{i}\eta} \tag{30a}$$

and

$$Y^{(2)\mathrm{CS}}(k) = \frac{k}{(\beta^2+k^2)}\left(\frac{\beta-\mathrm{i}k}{\beta+\mathrm{i}k}\right)^{\mathrm{i}\eta}\left(1 - \frac{\beta(\beta+k\eta)}{(\beta^2+k^2)}\right). \tag{30b}$$

In deriving (29a)-(30b) we have made use of the standard integrals [6]

$$\int_0^\infty \mathrm{e}^{-\lambda z} z^\nu \Phi(a, c; pz)\,\mathrm{d}z = \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}\, {}_2F_1(a, \nu+1; c; p/\lambda) \tag{31}$$

and

$$\int_0^\infty \mathrm{e}^{-ax} x^{s-1}\Psi(b, d; \mu x)\,\mathrm{d}x = \frac{\Gamma(s)\Gamma(1+s-d)}{a^s\Gamma(1+b+s-d)}\, {}_2F_1(b, s; 1+s+b-d; 1-\mu/a)$$
$$\mathrm{Re}\, s > 0 \qquad 1 + \mathrm{Re}\, s > \mathrm{Re}\, d. \tag{32}$$

For the elements of $\det_2 A^{\mathrm{CS}}(k)$ we obtained

$$A^{(1,1)\mathrm{CS}}(k) = 1 - \lambda^{(1)}\frac{1}{(1+\mathrm{i}\eta)(\alpha-\mathrm{i}k)^2}\left[(\alpha+\mathrm{i}k)^{-1}\left(\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)^{\mathrm{i}\eta} {}_2F_1\left(1, \mathrm{i}\eta; 2+\mathrm{i}\eta; \frac{\alpha+\mathrm{i}k}{\alpha-\mathrm{i}k}\right) - \frac{1}{2\alpha}\, {}_2F_1\left(1, \mathrm{i}\eta; 2+\mathrm{i}\eta; \left(\frac{\alpha+\mathrm{i}k}{\alpha-\mathrm{i}k}\right)^2\right)\right] \tag{33}$$

$$A^{(1,2)CS}(k) = -\lambda^{(2)}\left\{\frac{-\beta}{2(\beta^2+k^2)(\alpha+\beta)^2} + \frac{1}{(1+i\eta)(\beta-ik)(\alpha-ik)}\left(1-\frac{\beta(\beta+k\eta)}{(\beta^2+k^2)}\right)\right.$$
$$\times\left[(\beta+ik)^{-1}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}{}_2F_1\left(1, i\eta; 2+i\eta; \frac{\alpha+ik}{\alpha-ik}\right)\right.$$
$$\left.\left. -(\alpha+\beta)^{-1}{}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\alpha+ik)(\beta+ik)}{(\alpha-ik)(\beta-ik)}\right)\right]\right\} \tag{34a}$$

$$A^{(2,1)CS}(k) = -\lambda^{(1)}\left\{\frac{-\beta}{2(\beta^2+k^2)}\left[\frac{1}{(\alpha+\beta)^2} - \frac{1}{(\alpha^2+k^2)}\left(\frac{\alpha-ik}{\alpha+ik}\right)^{i\eta}\right]\right.$$
$$+\frac{1}{(1+i\eta)(\beta-ik)(\alpha-ik)}\left(1-\frac{\beta(\beta+k\eta)}{(\beta^2+k^2)}\right)$$
$$\times\left[(\alpha+ik)^{-1}\left(\frac{\alpha-ik}{\alpha+ik}\right)^{i\eta}{}_2F_1\left(1, i\eta; 2+i\eta; \frac{\beta+ik}{\beta-ik}\right)\right.$$
$$\left.\left. -(\alpha+\beta)^{-1}{}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\alpha+ik)(\beta+ik)}{(\alpha-ik)(\beta-ik)}\right)\right]\right\} \tag{34b}$$

and

$$A^{(2,2)CS}(k) = 1-\lambda^{(2)}\left\{\frac{-(\beta^2+3k^2-4k\eta\beta)}{16\beta(\beta^2+k^2)^2} + \frac{\beta(2\beta^2+k^2+k\beta\eta)}{2(\beta^2+k^2)^3}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}\right.$$
$$+\frac{1}{(1+i\eta)(\beta-ik)^2}\left(1-\frac{\beta(\beta+k\eta)(\beta^2+2k^2-k\beta\eta)}{(\beta^2+k^2)^2}\right)$$
$$\times\left[(\beta+ik)^{-1}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}{}_2F_1\left(1, i\eta; 2+i\eta; \frac{\beta+ik}{\beta-ik}\right)\right.$$
$$\left.\left. -\frac{1}{2\beta}{}_2F_1\left(1, i\eta; 2+i\eta; \left(\frac{\beta+ik}{\beta-ik}\right)^2\right)\right]\right\}. \tag{35}$$

The quantities $a^{(i,j)CS}(k)$ can be found from (33)-(35).

From (15) and (22) it is clear that in presenting results for $A^{(i,j)CS}(k)$ we had to deal with certain non-trivial indefinite integrals. Thus, it is of considerable interest to see how these results were obtained. We demonstrate this by calculating the expression in (33).

We have

$$A^{(1,1)CS}(k) = 1-\lambda^{(1)}\int_0^\infty\int_0^r dr\,dr'\,e^{-\alpha r}\,G^{C(R)}(r, r')e^{-\alpha r'}. \tag{36}$$

The Green function $G^{C(R)}(r, r')$ can be expressed in terms of the outgoing wave Green function $G^{C(+)}(r, r')$. This relation is given by

$$G^{C(R)}(r, r') = G^{C(+)}(r, r') - 2ikrr'\Gamma(1+i\eta)\,e^{ik(r+r')}\Phi(1+i\eta, 2; -2ikr)\Psi(1+i\eta, 2; -2ikr') \tag{37}$$

with

$$G^{C(+)}(r, r') = 2ikrr'\,e^{ik(r+r')}\Gamma(1+i\eta)\Phi(1+i\eta, 2; -2ikr_<)\Psi(1+i\eta, 2; -2ikr_>). \tag{38}$$

Here $r_<$ and $r_>$ stand for the smaller and greater values of r and r'. In terms of (37) we write the double integral in (36) as

$$\int_0^\infty \int_0^r \mathrm{d}r\,\mathrm{d}r'\,\mathrm{e}^{-\alpha(r+r')}\,G^{\mathrm{C(R)}}(r,r') = \int_0^\infty \int_0^\infty \mathrm{d}r\,\mathrm{d}r'\,\mathrm{e}^{-\alpha(r+r')}G^{\mathrm{C}(+)}(r,r')$$
$$-2\mathrm{i}krr'\Gamma(1+\mathrm{i}\eta)\int_0^\infty r\,\mathrm{e}^{-(\alpha-\mathrm{i}k)r}\,\Phi(1+\mathrm{i}\eta,2;-2\mathrm{i}kr)\,\mathrm{d}r'$$
$$\times\int_0^\infty r'\,\mathrm{e}^{-(\alpha-\mathrm{i}k)r'}\Psi(1+\mathrm{i}\eta,2;-2\mathrm{i}kr')\,\mathrm{d}r'. \tag{39}$$

In a recent paper [7] two of us have shown that

$$\int_0^\infty \int_0^\infty \mathrm{d}r\,\mathrm{d}r'\,\mathrm{e}^{-\alpha(r+r')}G^{\mathrm{C}(+)}(r,r')$$
$$= -\frac{1}{2\alpha(1+\mathrm{i}\eta)(\alpha-\mathrm{i}k)^2}\,{}_2F_1\left(1,\mathrm{i}\eta;2+\mathrm{i}\eta;\left(\frac{\alpha+\mathrm{i}k}{\alpha-\mathrm{i}k}\right)^2\right). \tag{40}$$

The single integrals in (39) can be found by the use of (31) and (32). These results in conjunction with (39) and (40) reduce (36) in the form given in (33).

The Fredholm determinant defined through (33)-(35) appears to be quite complicated. One would, therefore, like to make certain useful checks on the result for $\det A^{\mathrm{CS}}(k)$ before using it in further studies. For $\lambda^{(1)}=\lambda^{(2)}=0$, $\det_2 A^{\mathrm{CS}}(k)\to \det A^{\mathrm{C}}(k)=1$. This is as expected since the Fredholm determinant associated with the regular solution of a local potential is unity. We have also verified that $[\det_2 A^{\mathrm{CS}}(k)]_{\eta=0}$ is the Fredholm determinant for the Graz-I potential given elsewhere [8]. On very general grounds one knows that $\det_2 A^{\mathrm{CS}}(k)$ should be real. This is, however, not apparent from our expression. To demonstrate the reality of our result for $\det_2 A^{\mathrm{CS}}(k)$ we note the following.

The complex conjugate of the element $A^{(1,1)\mathrm{CS}}(k)$ is given by

$$[A^{(1,1)\mathrm{CS}}(k)]^* = 1-\frac{\lambda^{(1)}}{(1-\mathrm{i}\eta)(\alpha+\mathrm{i}k)^2}\left[(\alpha-\mathrm{i}k)^{-1}\left(\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)^{\mathrm{i}\eta}{}_2F_1\left(1,-\mathrm{i}\eta;2-\mathrm{i}\eta;\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)\right.$$
$$\left.-\frac{1}{2\alpha}\,{}_2F_1\left(1,-\mathrm{i}\eta;2-\mathrm{i}\eta;\left(\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)^2\right)\right]. \tag{41}$$

If the hypergeometric functions in (41) are transformed by the recurrence relation

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,1+a-c;1+a-b;1/z)$$
$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,1+b-c;1+b-a;1/z) \tag{42}$$

we get $[A^{(1,1)\mathrm{CS}}(k)]^* = A^{(1,1)\mathrm{CS}}(k)$, thereby proving the reality. Similarly, one can show that the other elements of $\det_2 A^{\mathrm{CS}}(k)$ are also real.

In the above we have constructed a closed-form expression for $\tan\delta^{\mathrm{CS}}(k)$ for the Coulomb plus Graz-I potential by using the algorithms of the phase-function method. It would, however, be of considerable interest to have an expression for $\tan\delta^{\mathrm{CS}}(k,r)$ in order to see how the phase function builds up to the phase shift. In appendix 1 we present the result for $\tan\delta^{\mathrm{CS}}(k,r)$ and show that $\lim_{r\to\infty}\tan\delta^{\mathrm{CS}}(k,r)=\tan\delta^{\mathrm{CS}}(k)$ as given in (27). In contrast to the result for $\tan \delta^{CS}(k)$, we could find only an infinite series representation for $\tan \delta^{CS}(k, r)$, which appears to be too heavy for straightforward numerical treatment.

The Graz-II potential [9] is an improvement over the original Graz-I potential and yields an accurate fit to all experimental data currently accepted for elastic proton-proton scattering for $l \leq 2$. In operator form the Graz-II potential is given by

$$V_l = |v_l^{(1)}\rangle \lambda_l^{(1)} \langle v_l^{(1)}| + |v_l^{(2)}\rangle \lambda_l^{(2)} \langle v_l^{(2)}| \tag{43}$$

with

$$\langle r|v_l^{(1)}\rangle = 2^{-l}(l!)^{-1} r^l \left[\exp(-\beta_l^{(11)} r) + \gamma_l^{(1)}\left(1 - \frac{\beta_l^{(12)}}{2(l+1)} r\right)\exp(-\beta_l^{(12)} r)\right] \tag{44}$$

and

$$\langle r|v_l^{(2)}\rangle = 2^{-l}(l!)^{-1} r^l \left[\left(1 - \frac{\beta_l^{(21)} r}{2(l+1)}\right)\exp(-\beta_l^{(21)} r) + \gamma_l^{(2)}\left(1 - \frac{(4l+7)\beta_l^{(22)} r}{4(l+1)(l+2)} + \frac{[\beta_l^{(22)} r]^2}{4(l+1)(l+2)}\right)\exp(-\beta_l^{(22)} r)\right]. \tag{45}$$

An analysis similar to that for the Graz-I potential also applies for the potential in (43). In appendix 2 we have presented the results for $\delta_l(k)$ for the Coulomb-distorted Graz-II potential.

4. $^1S_0$ proton–proton scattering phase shift

The $^1S_0$ state in p–p scattering represents the most important partial wave which requires an exact treatment of the Coulomb distortion. In (27) we have derived an expression for $\tan \delta^{CS}(k)$ for the Coulomb-distorted Graz-I (CDG) separable potential by absorbing the Coulomb part of the potential in the comparison functions of the phase method. This enables us to bypass the characteristic difficulties associated with the long-range nature of the Coulomb interaction and thus include the Coulomb effect rigorously. It will therefore be of some interest to study the influence of the Coulomb distortion on phase shifts $\delta^{CS}(k)$ by using our expression in (27). We note that our result for $\tan \delta^{CS}(k)$ is quite simple. The only non-elementary function which enters our formula is of the form ${}_2F_1(1, i\eta; 2+i\eta; z)$. This hypergeometric function can easily be evaluated by using its integral representation.

In the uncoupled $^1S_0$ channel the parameters of the Graz-I potential are $\lambda^{(1)} = -2.395\ \mathrm{fm}^{-3}$, $\lambda^{(2)} = 58.052\ \mathrm{fm}^{-3}$, $\alpha = 1.244\ \mathrm{fm}^{-1}$ and $\beta = 2.3601\ \mathrm{fm}^{-1}$. We have chosen to work with $(2k\eta)^{-1} = 28.80$ fm. This is the proton Bohr radius. For these parameters we have presented in table 1 the numerical values for $\delta^{G}(k)$ and $\delta^{CG}(k)$ as a function of laboratory energy E_{lab} between 1 and 40 MeV. Note that the results for the pure Graz phase shifts $\delta^{G}(k)$ have been obtained by turning off the Coulomb interaction in the numerical routine for generating CDG phase shifts $\delta^{CG}(k)$. As expected [10] the Couomb-distortion effect is predominant at low and moderate energies. In particular, the results for $\delta^{CG}(k)$ and the corresponding values of $\delta^{G}(k)$ differ significantly only for $E_{lab} \leq 20$ MeV.

For $\lambda^{(2)} = 0$ the form factors in (25) coincide with those of Yamaguchi [11]. The Yamaguchi potential with $\lambda^{(1)} = -2.405\ \mathrm{fm}^{-3}$ and $\alpha = 1.1\ \mathrm{fm}^{-1}$ provides a reasonable

Table 1. Phase shifts $\delta^S(k)$ and $\delta^{CS}(k)$ as a function of laboratory energy E_{lab}.

E_{lab} (MeV)	Phase shift (deg)					
	Present work				Mathelitsch and Plessas (1987)	
	$\delta^G(k)$	$\delta^{CG}(k)$	$\delta^Y(k)$	$\delta^{CY}(k)$	$\delta^Y(k)$	$\delta^{CY}(k)$
1	56.28	32.68	55.15	31.55	54.70	31.10
2	60.23	45.57	58.75	44.09	58.39	43.74
3	61.03	50.90	59.47	49.34	59.17	49.05
4	60.96	53.43	59.34	51.81	59.08	51.56
5	60.54	54.66	58.86	52.98	58.62	52.76
6	59.99	55.26	58.22	53.49	58.01	53.29
8	58.74	55.46	56.76	53.48	56.57	53.32
10	57.44	55.05	55.24	52.85	55.08	52.70
12	56.20	54.37	53.76	51.94	53.61	51.81
14	54.97	53.57	52.33	50.93		
16	53.81	52.71	50.97	49.87	50.84	49.75
18	52.69	51.81	49.68	48.80		
20	51.66	50.96	48.45	47.75	48.34	47.65
22	50.66	50.10	47.29	46.73		
24	49.69	49.24	46.18	45.73		
26	48.72	48.37	45.12	44.77		
28	47.78	47.51	44.11	43.84		
30	46.85	46.65	43.15	42.95	43.07	42.87
32	46.01	45.85	42.23	42.08		
34	45.16	45.06	41.35	41.25		
36	44.31	44.25	40.51	40.45		
38	43.49	43.46	39.70	39.67		
40	42.65	42.65	38.92	38.91	38.85	38.86

fit to p-p scattering data [12]. In table 1 we have also presented the results for pure Yamaguchi and Coulomb plus Yamaguchi phase shifts $\delta^Y(k)$ and $\delta^{CY}(k)$ respectively. While the numbers for $\delta^Y(k)$ and $\delta^{CY}(k)$ are systematically lower than the corresponding values for Graz and CDG potentials, they permit comparison with the work of Mathelitsch and Plessas [13] who made use of an R-matrix formalism to compute scattering phase shifts for the Coulomb-distorted Yamaguchi potential. The results of Mathelitsch and Plessas are given in table 1. The data of these authors are slightly different from our results for $\delta^Y(k)$ and $\delta^{CY}(k)$ because they have chosen to work with $\lambda^{(1)} = -1.529\ \mathrm{fm}^{-3}$.

The 'residual' phase written as $\delta_r = \delta^{CS} - \delta^S$ is a critical quantity for comparing different methods of calculating Coulomb effects. It is of interest to note that at a given energy the values of δ_r for the three sets of data presented in the table are almost in exact agreement. This implies that δ_r is not so sensitively dependent on the particular interactions as is the case for δ^{CS} and δ^S. By comparing our results for δ_r with those of Mathelitsch and Plessas it can be concluded that the phase function method developed here can serve as an alternative to the so-called R-matrix formalism for dealing with the Coulomb-nuclear problem.

In the above we have considered the $^1S_0$ scattering phase shifts induced by the Coulomb-distorted Graz-I potential. We have calculated similar results for the Graz-II case also. Our values are in exact agreement with those of Schweiger *et al* [9].

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Appendix 1. Phase function $\delta^{CS}(k, r)$ for the Coulomb plus Graz-I potential

From (24) the expression for $\tan \delta^{CS}(k, r)$ is obtained as

$$\tan \delta^{CS}(k, r) = -\sum_{i,j=1}^{2} \lambda^{(i)} a^{(i,j)CS}(k)\, Y^{(j)CS}(k)\, Y^{(i)CS}(k, r) \times\left(k|\mathscr{F}^{C}(k)|^2 \det_2 A^{CS}(k) + \sum_{i,j=1}^{2} \lambda^{(i)} a^{(i,j)CS}(k)\, Y^{(j)CS}(k) X^{(i)CS}(k, r)\right)^{-1} \tag{A1.1}$$

with

$$X^{(i)CS}(k, r) = \tfrac{1}{2}\int_0^r \mathrm{d}r'\, v^{(i)}(r')(\mathscr{F}^{C}(-k) f^{C}(k, r') + \mathscr{F}^{C}(k) f^{C}(-k, r')) \tag{A1.2}$$

and

$$Y^{(i)CS}(k, r) = k\int_0^r \mathrm{d}r'\, v^{(i)}(r')\phi^{C}(k, r'). \tag{A1.3}$$

Comparison of equations (23) (specialised for the Graz-I potential), (28), (A1.2) and (A1.3) shows that

$$\lim_{r\to\infty} X^{(i)CS}(k, r) = X^{(i)CS}(k) \qquad \lim_{r\to\infty} Y^{(i)CS}(k, r) = Y^{(i)CS}(k). \tag{A1.4}$$

We have presented the results for $X^{(i)CS}(k)$ and $Y^{(i)CS}(k)$ in (29) and (30). Other quantities which occur in (A1.1) have been given in (33)-(35).

We have found

$$\begin{aligned} X^{(1)CS}(k, r) &= e^{\pi\eta/2}\, \mathrm{Re}\left[\frac{\mathscr{F}^{C}(-k)}{(\alpha - \mathrm{i}k)\Gamma(2+\mathrm{i}\eta)}\, {}_2F_1\left(1, \mathrm{i}\eta; 2+\mathrm{i}\eta; \frac{\alpha+\mathrm{i}k}{\alpha-\mathrm{i}k}\right)\right] \\ &\quad + e^{\pi\eta/2}\, \mathrm{Re}\left\{\frac{\mathscr{F}^{C}(-k)2\mathrm{i}k\pi}{\Gamma(\mathrm{i}\eta)\Gamma(1+\mathrm{i}\eta)\sin(2\pi)}\, e^{-(\alpha-\mathrm{i}k)r}\right. \\ &\quad \times\left[\frac{\Gamma(1+\mathrm{i}\eta)}{(\alpha^2+k^2)}\left(\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)^{\mathrm{i}\eta}[1+(\alpha-\mathrm{i}k)r]\right. \\ &\quad + r^2\sum_{n=1}^{\infty}\sum_{s=0}^{n-1}\frac{\Gamma(n+1+\mathrm{i}\eta)}{n!}\left(-\frac{2\mathrm{i}k}{\alpha-\mathrm{i}k}\right)^{n}\frac{[(\alpha-\mathrm{i}k)r]^{s}}{(s+2)!} \\ &\quad \left.\left. + \frac{1}{2\mathrm{i}k(\alpha-\mathrm{i}k)}\sum_{n=0}^{\infty}\sum_{s=0}^{n-1}\frac{\Gamma(n+\mathrm{i}\eta)}{\Gamma(n)}\left(-\frac{2\mathrm{i}k}{\alpha-\mathrm{i}k}\right)^{n}\frac{[(\alpha-\mathrm{i}k)r]^{s}}{s!}\right]\right\} \end{aligned} \tag{A1.5}$$

$$X^{(2)\mathrm{CS}}(k,r)=X^{(1)\mathrm{CS}}(k,r)|_{\alpha=\beta}+\frac{\beta}{2}\mathrm{e}^{\pi\eta/2}\,\mathrm{Re}\left\{\frac{\mathscr{F}^{\mathrm{C}}(-k)}{(\beta-\mathrm{i}k)\Gamma(2+\mathrm{i}\eta)}\right.$$

$$\left.\times\left[\frac{1+\mathrm{i}\eta}{(\beta+\mathrm{i}k)}-\frac{2(\beta+k\eta)}{(\beta^2+k^2)}\,{}_2F_1\left(1,\mathrm{i}\eta;2+\mathrm{i}\eta;\frac{\beta+\mathrm{i}k}{\beta-\mathrm{i}k}\right)\right]\right\}$$

$$+\frac{\beta}{2}\mathrm{e}^{\pi\eta/2}\,\mathrm{Re}\left\{\frac{\mathscr{F}^{\mathrm{C}}(-k)2\mathrm{i}k\pi}{\Gamma(\mathrm{i}\eta)\Gamma(1+\mathrm{i}\eta)\sin(2\pi)}\mathrm{e}^{-(\beta-\mathrm{i}k)r}\right.$$

$$\times\left[-\frac{\Gamma(1+\mathrm{i}\eta)}{(\beta^2+k^2)}\left(\frac{\beta-\mathrm{i}k}{\beta+\mathrm{i}k}\right)^{\mathrm{i}\eta}\left((\beta-\mathrm{i}k)r^2+\frac{2(\beta+k\eta)(1+\beta-\mathrm{i}k)r}{(\beta^2+k^2)}\right)\right.$$

$$+r^2\sum_{n=1}^{\infty}\sum_{s=0}^{n-1}\frac{\Gamma(n+1+\mathrm{i}\eta)}{n!}\left(-\frac{2\mathrm{i}k}{\beta-\mathrm{i}k}\right)^n\frac{[(\beta-\mathrm{i}k)r]^s}{(s+2)!}\left(\frac{(s-n)}{(\beta-\mathrm{i}k)}-r\right)$$

$$+\frac{1}{2\mathrm{i}k(\beta-\mathrm{i}k)}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{\Gamma(n+\mathrm{i}\eta)}{\Gamma(n)}\left(-\frac{2\mathrm{i}k}{\beta-\mathrm{i}k}\right)^n\frac{[(\beta-\mathrm{i}k)r]^s}{s!}$$

$$\left.\left.\times\left(\frac{(s-n-1)}{(\beta-\mathrm{i}k)}-r\right)\right]\right\}\qquad(\mathrm{A}1.6)$$

$$Y^{(1)\mathrm{CS}}(k,r)=\frac{k}{(\alpha^2+k^2)}\left(\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)^{\mathrm{i}\eta}\{1-\mathrm{e}^{-(\alpha-\mathrm{i}k)r}[1+(\alpha-\mathrm{i}k)r]\}$$

$$-\frac{kr^2}{\Gamma(1+\mathrm{i}\eta)}\mathrm{e}^{-(\alpha-\mathrm{i}k)r}\sum_{n=1}^{\infty}\sum_{s=0}^{n-1}\frac{\Gamma(n+1+\mathrm{i}\eta)}{n!}\left(-\frac{2\mathrm{i}k}{\alpha-\mathrm{i}k}\right)^n\frac{[(\alpha-\mathrm{i}k)r]^S}{(s+2)!}\qquad(\mathrm{A}1.7)$$

and

$$Y^{(2)\mathrm{CS}}(k,r)=Y^{(1)\mathrm{CS}}(k,r)|_{\alpha=\beta}+\frac{k\beta(\beta+k\eta)}{(\beta^2+k^2)^2}\left(\frac{\beta-\mathrm{i}k}{\beta+\mathrm{i}k}\right)^{\mathrm{i}\eta}$$

$$+\frac{k\beta}{2}\mathrm{e}^{-(\beta-\mathrm{i}k)r}\left[\frac{1}{(\beta^2+k^2)^2}\left(\frac{\beta-\mathrm{i}k}{\beta+\mathrm{i}k}\right)^{\mathrm{i}\eta}\{[1+(\beta-\mathrm{i}k)r]\right.$$

$$\times[2(\beta+k\eta)+r(\beta^2+k^2)]+(\beta^2+k^2)(\beta-\mathrm{i}k)^2r^3\}-\frac{r^3}{\Gamma(1+\mathrm{i}\eta)}$$

$$\left.\times\sum_{n=1}^{\infty}\sum_{s=0}^{n-1}\frac{\Gamma(n+1+\mathrm{i}\eta)}{n!}\left(-\frac{2\mathrm{i}k}{\beta-\mathrm{i}k}\right)^n\frac{[(\beta-\mathrm{i}k)r]^s}{(s+2)!}[s-n-(\beta-\mathrm{i}k)r]\right].\qquad(\mathrm{A}1.8)$$

It is easy to see that in the limit $r\to\infty$ (A1.5)-(A1.8) yield the results in (29) and (30). This serves as a useful check on our results for $X^{(i)\mathrm{CS}}(k,r)$ and $Y^{(i)\mathrm{CS}}(k,r)$. Thus (A1.1) together with (29)-(35) and (A1.5)-(A1.8) give our desired expression for $\tan\delta^{\mathrm{CS}}(k,r)$.

Appendix 2. $\delta_l^{\mathrm{CS}}(k)$ for the Coulomb plus Graz-II potential

The form factors in (44) and (45) can be rewritten as

$$\langle r|v_l^{(1)}\rangle = 2^{-l}(l!)^{-1}r^l\bigg[\exp(-\beta_l^{(11)}r) + \gamma_l^{(1)}\bigg(\exp(-\beta_l^{(12)}r)+\frac{\beta_l^{(12)}}{2(l+1)}\frac{\partial}{\partial\beta_l^{(12)}}\exp(-\beta_l^{(12)}r)\bigg)\bigg] \tag{A2.1}$$

and

$$\langle r|v_l^{(2)}\rangle = 2^{-l}(l!)^{-1}r^l\bigg[\exp(-\beta_l^{(21)}r)+\frac{\beta_l^{(21)}}{2(l+1)}\frac{\partial}{\partial\beta_l^{(21)}}\exp(-\beta_l^{(21)}r) + \gamma_l^{(2)}\bigg(\exp(-\beta_l^{(22)}r)+\frac{(4l+7)\beta_l^{(22)}}{4(l+1)(l+2)}\frac{\partial}{\partial\beta_l^{(22)}}\exp(-\beta_l^{(22)}r) + \frac{(\beta_l^{(22)}r)^2}{4(l+1)(l+2)}\frac{\partial}{\partial(\beta_l^{22})^2}\exp(-\beta_l^{(22)}r)\bigg)\bigg]. \tag{A2.2}$$

For this type of form factors it is also possible to derive an analytical formula for $\tan\delta_l^{\mathrm{CS}}(k)$. From (24) we see that the quantities of interest are

$$X_l^{(i)\mathrm{CS}}(k) = \mathrm{Re}\bigg(\mathscr{F}_l^{\mathrm{C}}(-k)\int_0^\infty \mathrm{d}r\, v_l^{(i)}(r) f_l^{\mathrm{C}}(k,r)\bigg) \tag{A2.3}$$

$$Y_l^{(i)\mathrm{CS}}(k) = k\int_0^\infty \mathrm{d}r\, v_l^{(i)}(r)\phi_l^{\mathrm{C}}(k,r) \tag{A2.4}$$

and

$$A_l^{(i,j)\mathrm{CS}}(k) = \delta_{ij} - \lambda_l^{(j)}\int_0^\infty\int_0^r \mathrm{d}r\,\mathrm{d}r'\, G_l^{\mathrm{C(R)}}(r,r')v_l^{(j)}(r)v_l^{(i)}(r') \qquad i,j=1,2. \tag{A2.5}$$

In view of (A2.1) and (A2.2) it is easy to see that the results for $X_l^{(i)\mathrm{CS}}(k)$ and $Y_l^{(i)\mathrm{CS}}(k)$ can be written in terms of the basic integrals

$$\int_0^\infty \mathrm{d}r\, r^l\, \mathrm{e}^{-\beta r} f_l^{\mathrm{C}}(k,r)$$

and

$$\int_0^\infty \mathrm{d}r\, r^l\, \mathrm{e}^{-\beta r} \phi_l^{\mathrm{C}}(k,r)$$

and their appropriate derivatives with respect to β. For these integrals we have found

$$\int_0^\infty \mathrm{d}r\, r^l\, \mathrm{e}^{-\beta r} f_l^{\mathrm{C}}(k,r) = \frac{\mathrm{e}^{\pi\eta/2}\Gamma(2l+2)}{(2\mathrm{i}k)^l\Gamma(l+2+\mathrm{i}\eta)(\beta-\mathrm{i}k)}\,{}_2F_1\bigg(1,\mathrm{i}\eta-l;\, l+2+\mathrm{i}\eta;\, \frac{\beta+\mathrm{i}k}{\beta-\mathrm{i}k}\bigg) \tag{A2.6}$$

and

$$\int_0^\infty \mathrm{d}r\, r^l\, \mathrm{e}^{-\beta r} \phi_l^{\mathrm{C}}(k,r) = \frac{\Gamma(2l+2)}{(\beta^2+k^2)^{l+1}}\bigg(\frac{\beta-\mathrm{i}k}{\beta+\mathrm{i}k}\bigg)^{\mathrm{i}\eta}. \tag{A2.7}$$

The double integral in (A2.5) can be evaluated from the knowledge of

$$\bar{G}_l^{C(R)}(\alpha, \beta) = \int_0^\infty \int_0^r dr\, dr'\, (rr')^l\, e^{-\alpha r}\, G_l^{C(R)}(r, r')\, e^{-\beta r'}$$

$$= \frac{\Gamma(2l+2)}{(l+1+i\eta)(\alpha - ik)(\beta - ik)} \left[\frac{1}{(\alpha^2+k^2)^l(\alpha+ik)} \left(\frac{\alpha - ik}{\alpha + ik}\right)^{i\eta} \right.$$

$$\times {}_2F_1\left(1, i\eta - l; l+2+i\eta; \frac{\beta+ik}{\beta-ik}\right) - (\alpha+\beta)^{-2l-1}$$

$$\left. \times {}_2F_1\left(1, i\eta - l; l+2+i\eta; \frac{(\alpha+ik)(\beta+ik)}{(\alpha-ik)(\beta-ik)}\right)\right]. \tag{A2.8}$$

The following derivatives will also be needed:

$$\frac{\partial}{\partial\beta} \bar{G}_l^{C(R)}(\alpha, \beta) = \frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)}{(\beta^2+k^2)} \left[\frac{1}{(\alpha^2+k^2)^{l+1}} \left(\frac{\alpha - ik}{\alpha + ik}\right)^{i\eta} - \frac{1}{(\alpha+\beta)^{2l+2}} \right]$$

$$- \frac{2[(l+1)\beta + k\eta]}{(\beta^2+k^2)} \bar{G}_l^{C(R)}(\alpha, \beta) \tag{A2.9}$$

$$\frac{\partial^2}{\partial\alpha\partial\beta} \bar{G}_l^{C(R)}(\alpha, \beta)$$

$$= \frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)}{(\alpha^2+k^2)(\beta^2+k^2)} \left[\frac{2(l+1)(\alpha^2+\beta^2+k^2+\alpha\beta) + 2k\eta(\alpha+\beta)}{(\alpha+\beta)^{2l+3}} \right.$$

$$\left. - \frac{2[(l+1)\alpha + k\eta]}{(\alpha^2+k^2)^{l+1}} \left(\frac{\alpha - ik}{\alpha + ik}\right)^{i\eta} \right]$$

$$+ \frac{4[(l+1)\beta + k\eta][(l+1)\alpha + k\eta]}{(\alpha^2+k^2)(\beta^2+k^2)} \bar{G}_l^{C(R)}(\alpha, \beta) \tag{A2.10}$$

$$\frac{\partial^3}{\partial\alpha\partial\beta^2} \bar{G}_l^{C(R)}(\alpha, \beta)$$

$$= \frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)}{(\alpha^2+k^2)(\beta^2+k^2)} \left[\frac{2[(l+1)(\alpha+2\beta) + k\eta](\beta^2+k^2)}{(\alpha+\beta)^{2l+3}} \right.$$

$$- \frac{4[(l+1)\beta + k\eta][(l+1)\alpha + k\eta](\beta^2+k^2)}{(\alpha+\beta)^{2l+4}}$$

$$- \frac{2[(l+1)(\alpha^2+\beta^2+k^2+\alpha\beta) + k\eta(\alpha+\beta)}{(\alpha+\beta)^{2l+4}} [(2l+5)\beta^2 + (2l+3)k^2 + 2\alpha\beta]$$

$$\left. + \frac{4[(l+1)\alpha + k\eta][(l+1)\beta + k\eta]}{(\alpha^2+k^2)^{l+1}} \left(\frac{\alpha - ik}{\alpha + ik}\right)^{i\eta} \right]$$

$$+ \frac{4[(l+1)\alpha + k\eta]\{(\beta^2+k^2)(l+1) - 2[(l+1)\beta + k\eta][(l+2)\beta + k\eta]\}}{(\alpha^2+k^2)(\beta^2+k^2)^2}$$

$$\times \bar{G}_l^{C(R)}(\alpha, \beta) \tag{A2.11}$$

and

$$
\begin{aligned}
&\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2}\bar{G}_l^{C(R)}(\alpha,\beta)\\
&\quad=\frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)}{(\alpha^2+k^2)^2(\beta^2+k^2)^2}\left[\frac{2(l+1)(\alpha^2+k^2)(\beta^2+k^2)[\alpha-2k\eta-(2l+1)\beta]}{(\alpha+\beta)^{2l+4}}\right.\\
&\quad-\frac{2(\beta^2+k^2)[(l+1)(\alpha+2\beta)+k\eta][(2l+5)\alpha^2+(2l+3)k^2+2\alpha\beta]}{(\alpha+\beta)^{2l+4}}\\
&\quad+\frac{8(\beta^2+k^2)}{(\alpha+\beta)^{2l+5}}[(l+1)\alpha+k\eta][(l+1)\beta+k\eta][(l+3)\alpha^2+(l+2)k^2+\alpha\beta]\\
&\quad-\frac{2[(l+1)(2\alpha+\beta)+k\eta](\alpha^2+k^2)}{(\alpha+\beta)^{2l+4}}[(2l+5)\beta^2+(2l+3)k^2+2\alpha\beta]\\
&\quad-\frac{4[(l+1)(\alpha^2+\beta^2+k^2+\alpha\beta)+k\eta(\alpha+\beta)]}{(\alpha+\beta)^{2l+5}}\{\beta(\alpha+\beta)(\alpha^2+k^2)\\
&\quad-[(2l+5)\beta^2+(2l+3)k^2+2\alpha\beta]\\
&\quad\times[(l+3)\alpha^2+(l+2)k^2+\alpha\beta]\}\\
&\quad+\frac{4[(l+1)\beta+k\eta]\{(l+1)(\alpha^2+k^2)-2[(l+1)\alpha+k\eta][(l+2)\alpha+k\eta]\}}{(\alpha^2+k^2)^{l+1}}\\
&\quad\times\left(\frac{\alpha-\mathrm{i}k}{\alpha+\mathrm{i}k}\right)^{\mathrm{i}\eta}\\
&\quad\left.-\frac{4[(l+1)\alpha+k\eta]\{(l+1)(\beta^2+k^2)-[(l+1)\beta+k\eta][(l+2)\beta+k\eta]\}}{(\alpha+\beta)^{2l+2}}\right]\\
&\quad+\frac{4\{(l+1)(\beta^2+k^2)-[(l+1)\beta+k\eta][(l+2)\beta+k\eta]\}}{(\alpha^2+k^2)^2(\beta^2+k^2)^2}\\
&\quad\times\{(l+1)(\alpha^2+k^2)-[(l+1)\alpha+k\eta][(l+2)\alpha+k\eta]\}\bar{G}_l^{C(R)}(\alpha,\beta).
\end{aligned}
\tag{A2.12}
$$

With the single and double integrals given above we have expressed $\tan\delta_l^{CS}(k)$ for the Coulomb plus Graz-II potential in closed analytic form.

References

[1] Calogero F 1967 *Variable Phase Approach to Potential Scattering* (New York: Academic)
[2] Fano U, Theodosiou C E and Dehmer J L 1976 *Rev. Mod. Phys.* **48** 49
[3] Talukdar B, Chattarji D and Banerjee P 1977 *J. Phys. G: Nucl. Phys.* **3** 813
[4] van Haeringen H and van Wageningen R 1975 *J. Math. Phys.* **16** 1441
[5] Crepinsek L, Lang C B, Oberhumer H, Plessas W and Zingl H 1975 *Acta. Phys. Austr.* **42** 139
[6] Slater L J 1960 *Confluent Hypergeometric Functions* (Cambridge: Cambridge University Press)
[7] Talukdar B, Laha U and Bhattaru S R 1985 *J. Phys. A: Math. Gen.* **18** L359
[8] Talukdar B, Saha S and Sasakawa T 1983 *J. Math. Phys.* **24** 683
[9] Schweiger W, Plessas W, Kok L P and van Haeringen H 1983 *Phys. Rev.* C **27** 515
[10] Plessas W, Streit L and Zingl H 1974 *Acta Phys. Austr.* **40** 272
[11] Yamaguchi Y 1954 *Phys. Rev.* **95** 1628
[12] van Haeringen H 1975 *Nucl. Phys.* A **253** 355
[13] Mathelitsch L and Plessas W 1987 *Proc. 8th Int. Conf. on Few-body Systems and Nuclear Forces, Graz* (*Lecture Notes in Physics* **87**) (Berlin: Springer)